

# THE NASH-MOSER THEOREM OF HAMILTON AND RIGIDITY OF FINITE DIMENSIONAL NILPOTENT LIE ALGEBRAS

ALFREDO BREGA, LEANDRO CAGLIERO, AND AUGUSTO CHAVES OCHOA

**ABSTRACT.** We apply the Nash-Moser theorem for exact sequences of R. Hamilton to the context of deformations of Lie algebras and we discuss some aspects of the scope of this theorem in connection with the polynomial ideal associated to the variety of nilpotent Lie algebras. This allows us to introduce the space  $H_{k-nil}^2(\mathfrak{g}, \mathfrak{g})$ , and certain subspaces of it, that provide fine information about the deformations of  $\mathfrak{g}$  in the variety of  $k$ -step nilpotent Lie algebras. We also include an elementary proof of a finite dimensional version of the Nash-Moser theorem for exact sequences of R. Hamilton.

Then we focus on degenerations and rigidity in the variety of  $k$ -step nilpotent Lie algebras of dimension  $n$  with  $n \leq 7$  and, in particular, we obtain rigid Lie algebras and rigid curves in the variety of 3-step nilpotent Lie algebras of dimension 7. We also recover some known results and point out some possible errors in the bibliography related to the classification and deformations of these Lie algebras.

## 1. INTRODUCTION

In this paper we will assume that all Lie algebras and representations are finite dimensional, and mostly over  $\mathbb{R}$ .

In the first part of the paper we give an elementary proof of a finite dimensional version of the Nash-Moser theorem for exact sequences of R. Hamilton. In the second part, we apply this theorem to the context of deformations in the variety of nilpotent Lie algebras. Our main results are described below.

**1.1. The Nash-Moser theorem of R. Hamilton.** A very well known general principle of deformation theory says that given an (algebraic) structure  $\mu$ , then

$$(1.1) \quad H^2(\mu, \mu) = 0 \Rightarrow \mu \text{ is rigid, but the converse is not true in general.}$$

By definition, an algebraic structure  $\mu$  on a  $\mathbb{K}$ -vector space  $V$  is rigid if the  $GL(V)$ -orbit of  $\mu$ ,  $\mathcal{O}(\mu)$ , is a Zariski open set in the algebraic variety of all such algebraic structures.

When  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ , an algebraic structure  $\mu$  is rigid, roughly speaking, if every small perturbation of  $\mu$  is isomorphic to  $\mu$ . More precisely, it is known that  $\mathcal{O}(\mu)$  is open in the metric topology if and only if it is open in the Zariski topology (see [NR, Proposition 17.1], see also [GK, Proposition 2]). As a consequence of this, the principle (1.1) follows from a particular instance of the Nash-Moser theorem for exact sequences of R. Hamilton as

---

Partially supported by SECyT-UNC, FONCyT and CONICET grants.

we recall below. This theorem is stated in [H] in terms of tame Fréchet spaces and it is related to the inverse function theorem of Nash and Moser [Na, Mo]. A finite dimensional version of it is the following.

**Theorem 1.1.** *Let  $U_i \in \mathbb{R}^{n_i}$  ( $i = 1, 2, 3$ ) be open sets and let*

$$(1.2) \quad U_1 \xrightarrow{F} U_2 \xrightarrow{G} U_3$$

*be a sequence of  $C^\infty$  functions such that  $G \circ F$  is constant, say  $G(F(x_1)) = c \in U_3$  for all  $x_1 \in U_1$ . If the linear sequence*

$$(1.3) \quad T_a U_1 \xrightarrow{dF|_a} T_{F(a)} U_2 \xrightarrow{dG|_{F(a)}} T_c U_3$$

*is exact (in the usual algebraic sense) for certain  $a \in U_1$ , then the sequence (1.2) “locally splits”, that is: there is a neighborhood  $U'_2 \subset U_2$  of  $F(a)$  such that for every  $x_2 \in U'_2$  satisfying  $G(x_2) = c$  there exists  $x_1 \in U_1$ , that depends smoothly on  $x_2$ , such that  $F(x_1) = x_2$ .*

We could not find this statement in the literature. Recently, I. Struchiner pointed out to us that a close result appears in Serre’s book (see in [S, pp. 89-90] the result attributed to Weil). Also, and remarkably to us, we do not see this theorem frequently cited in articles dealing with algebraic structures in the context of (1.1).

Hamilton’s proof of this result, in the context of tame Fréchet spaces and tame smooth functions, is considerably involved. We came across Hamilton’s paper because it is cited in the survey [CSS] of M. Crainic, F. Schätz and I. Struchiner where the authors address, in a unified way, several well known problems about rigidity and stability of Lie algebras and morphisms based on the principle (1.1). To address these problems, the authors state and prove some stability results (see Propositions 4.3, 4.4 and 4.5 in [CSS]) that are phrased in terms of Kuranishi models and non-degenerate zeros of equivariant sections of vector bundles with group actions.

However, we think that both, the Nash-Moser theorem of Hamilton for tame Fréchet spaces and the results about Kuranishi models and equivariant sections of [CSS], are much deeper than what is needed to address some problems about rigidity and stability in a finite dimensional context.

Our main contribution in the first part of the paper is to give a precise statement of the Nash-Moser theorem of Hamilton for  $\mathbb{R}^n$  (see Theorem 2.3 below) with an elementary proof of it, giving also a brief discussion of the scope of this theorem. We hope that this will help to make this classic and important result accessible to more people working on deformations of algebraic structures.

**1.2. Degenerations and rigidity of nilpotent Lie algebras.** Theorem 1.1 can be applied to the study of the deformations of any algebraic structure  $\mu$  on a finite dimensional  $\mathbb{R}$ -vector space  $V$  and, in particular, when  $\mu$  defines a Lie algebra structure on  $V$ .

Let  $\mathcal{L}_n$  (resp.  $\mathcal{N}_n$ ) be the algebraic variety of all Lie algebra (resp. nilpotent Lie algebra) structures  $\mu$  on an  $n$ -dimensional vector space  $\mathfrak{g}$ . Let  $\{\mathfrak{g}^i\}_{i \geq 0}$  and  $\{\mathfrak{g}^{(i)}\}_{i \geq 0}$  denote, respectively, the descending central series and the derived series of  $\mathfrak{g}$  (i.e.  $\mathfrak{g}^0 = \mathfrak{g}^{(0)} = \mathfrak{g}$ ,  $\mathfrak{g}^i = [\mathfrak{g}^{i-1}, \mathfrak{g}]$  and  $\mathfrak{g}^{(i)} = [\mathfrak{g}^{(i-1)}, \mathfrak{g}^{(i-1)}]$ ) and let  $\mathcal{N}_{n,k} = \{\mathfrak{g} \in \mathcal{N}_n : \mathfrak{g}^k = 0\}$  and  $\mathcal{S}_{n,k} = \{\mathfrak{g} \in \mathcal{L}_n :$

$\mathfrak{g}^{(k)} = 0\}$  be, respectively, the subvariety of all (at most)  $k$ -step nilpotent and solvable Lie algebras.

Theorem 1.1 applies to the study of the deformations of  $\mu \in \mathcal{L}_n$  considering the following  $C^\infty$  functions,

$$(1.4) \quad GL(\mathfrak{g}) \xrightarrow{F} \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g} \xrightarrow{G=J} \Lambda^3 \mathfrak{g}^* \otimes \mathfrak{g}$$

where  $F$  is the action of  $GL(V)$  on  $\mu$  and  $J$  is the Jacobi operator, that is

$$\begin{aligned} F(g)(x, y) &= g(\mu(g^{-1}x, g^{-1}y)), \quad g \in GL(\mathfrak{g}); \\ J(\sigma)(x, y, z) &= \sum_{cyclic} \sigma(\sigma(x, y), z), \quad \sigma \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g} \quad \text{and} \quad x, y, z \in \mathfrak{g}. \end{aligned}$$

On the other hand, if we are interested in the deformations of  $\mu \in \mathcal{N}_{n,k}$  we can apply Theorem 1.1 considering the following  $C^\infty$  functions,

$$(1.5) \quad GL(\mathfrak{g}) \xrightarrow{F} \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g} \xrightarrow{G=J \oplus N_k} \Lambda^3 \mathfrak{g}^* \otimes \mathfrak{g} \oplus (\mathfrak{g}^*)^{\otimes(k+1)} \otimes \mathfrak{g},$$

where  $F$  is as above and  $N_k : \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g} \rightarrow (\mathfrak{g}^*)^{\otimes(k+1)} \otimes \mathfrak{g}$  is given by,

$$(1.6) \quad N_k(\sigma)(x_1, \dots, x_{k+1}) = \sigma(\dots \sigma(\sigma(x_1, x_2), x_3), \dots, x_{k+1}) \quad \text{for } k \geq 1.$$

This sequence allows us to introduce the cohomology space  $H_{k-nil}^2(\mathfrak{g}, \mathfrak{g})$ .

Theorem 1.1 could be applied in a more subtle way. Note that  $J$  and  $N_k$  give rise, when written in coordinates, to polynomials of degree 2 and  $k$  respectively. Let  $I_{n,k}$  be the ideal generated by these polynomials. It turns out that, depending on  $n$  and  $k$ , there might be polynomials of degree less than  $k$  in  $\sqrt{I_{n,k}}$  that are not in  $I_{n,k}$ . In this case, if  $P$  is such a polynomial, it can be used as (part of) the function  $G$  in Theorem 1.1 to describe more precisely  $\mathcal{N}_{n,k}$  which, in turn, might help to recognize rigid  $\mu$ 's as points with “zero cohomology”. This happens for the ideal of  $I_{7,6}$  (which defines  $\mathcal{N}_7 = \mathcal{N}_{7,6}$ ). Indeed, the polynomial identity  $[\mathfrak{g}^1, \mathfrak{g}^3] = 0$ , of degree 5, holds for every nilpotent Lie algebra  $\mathfrak{g}$  of dimension 7. This is discussed in §3.3 and, in §4.3, we use the identity  $[\mathfrak{g}^1, \mathfrak{g}^3] = 0$  and Theorem 1.1 to recover the result that states that the only three (two over  $\mathbb{C}$ ) curves in  $\mathcal{N}_7$  are rigid curves. As a byproduct we obtain a curve, consisting of solvable Lie algebras, that is rigid in the variety of Lie algebras satisfying  $[\mathfrak{g}^1, \mathfrak{g}^3] = 0$ .

The paper also includes an analysis of all rigid Lie algebras in  $\mathcal{N}_{n,k}$  for  $n = 5, 6$ . Some of these results provide examples showing that the converse part of the principle (1.1) is false in  $\mathcal{N}_{n,k}$  (in analogy with the famous example of Richardson [R] in  $\mathcal{L}_{18}$ ).

Finally, in §4.4, we discuss degenerations and rigidity in  $\mathcal{N}_{7,3}$ . As far as we know, the results of this subsection are new. We obtain three rigid Lie algebras and three (two over  $\mathbb{C}$ ) rigid curves in  $\mathcal{N}_{7,3}$ . We also present degenerations for all Lie algebras  $\mathfrak{g} \in \mathcal{N}_{7,3}$  with  $\dim H_{3-nil}^2(\mathfrak{g}, \mathfrak{g}) = 1$  and we point out some possible errors in the bibliography. In particular, we provide a non-trivial deformation of a Lie algebra in  $\mathcal{N}_{7,3}$  which is claimed to be rigid in [GR].

## 2. THE NASH-MOSER THEOREM FOR EXACT SEQUENCES OF HAMILTON

**2.1. Hamilton's statement of the theorem.** In [H, Theorem 3.1.1], R. Hamilton proves a theorem that he calls *The Nash-Moser theorem for exact sequences*, see Theorem 2.1 below. This theorem is a generalization of the inverse function theorem of Nash and Moser [Na, Mo] that Hamilton finds it “useful in problems involving deformation of structures”.

Let  $\mathcal{F}_1$ ,  $\mathcal{F}_2$  and  $\mathcal{F}_3$  be tame Fréchet spaces, let  $U_i \subset \mathcal{F}_i$  ( $i = 1, 2, 3$ ) be open subsets and let

$$(2.1) \quad U_1 \xrightarrow{F} U_2 \xrightarrow{G} U_3$$

be two smooth tame maps such that  $G(F(x_1)) = 0 \in U_3$  for all  $x_1 \in U_1$ . Thus  $\text{Im } F \subset \{G = 0\}$ , where  $\{G = 0\} \subset U_2$  denotes the level set  $\{x_2 \in U_2 : G(x_2) = 0\}$ . We will refer to the sequence (2.1) as a *3-term  $C^\infty$ -chain complex*. We say that this sequence *locally splits* at  $x_2 = F(x_1) \in U_2$  ( $x_1 \in U_1$ ) if there are neighborhoods  $U'_2 \subset U_2$  of  $x_2$  and  $U'_1 \subset U_1$  of  $x_1$ , and a smooth tame map  $H$  as follows

$$\begin{array}{ccccc} U_1 & \xrightarrow{F} & U_2 & \xrightarrow{G} & U_3 \\ \uparrow & & \uparrow & & \\ U'_1 & \xleftarrow{H} & U'_2 & & \end{array}$$

such that

$$(F \circ H)|_{U'_2 \cap \{G=0\}} = \text{Id}|_{U'_2 \cap \{G=0\}}.$$

Looking at the corresponding tangent spaces, the following natural observations arise:

- (1)  *$C^\infty$ -chain  $\Leftrightarrow$  ordinary chain*: It is clear that a 3-term  $C^\infty$ -chain complex induces, for each  $x_1 \in U_1$ , a 3-term (ordinary) chain complex in the corresponding tangent spaces,

$$(2.2) \quad T_{x_1}U_1 \xrightarrow{dF|_{x_1}} T_{F(x_1)}U_2 \xrightarrow{dG|_{F(x_1)}} T_{x_3}U_3.$$

On the other hand, if  $U_1$  is connected and (2.2) is a chain complex of linear spaces for all  $x_1 \in U_1$ , then  $U_1 \xrightarrow{F} U_2 \xrightarrow{G} U_3$  is a 3-term  $C^\infty$ -chain complex. It is obvious that it is not sufficient to require that (2.2) be a chain complex just for one  $x_1 \in U_1$ .

- (2) *local splitting  $\nRightarrow$  linear splitting*: An unpleasant example is  $F(t) = (t, 0)$  and  $G(x, y) = y^2$  which is obtained with a “bad”  $G$ . We point out that *local exactness  $\nRightarrow$  linear splitting* even with a “not-so-bad”  $G$ , for instance consider the cases of a cusp  $F(t) = (t^2, t^3)$  and  $G(x, y) = x^3 - y^2$ ; or the double point  $F(t) = (t^2 - 1, t(t^2 - 1))$  and  $G(x, y) = x^3 + x^2 - y^2$  (note that these two  $G$ 's are irreducible polynomials).

It is not clear to us how “good” must be  $G$  (in the finite dimensional case) so that *local splitting and “good”  $G \Rightarrow$  linear splitting*.

- (3) *linear splitting  $\Rightarrow$  local splitting*: This is the content of the Nash-Moser theorem for exact sequences: it roughly states that if the 3-term chain complex (2.2), induced by  $U_1 \xrightarrow{F} U_2 \xrightarrow{G} U_3$ , is exact (and thus, it splits), then  $U_1 \xrightarrow{F} U_2 \xrightarrow{G} U_3$  locally splits at  $F(x_1)$ .

We now state the Nash-Moser theorem for exact sequences of Hamilton basically in the same terms as in it appears in [H].

**Theorem 2.1.** *Let  $U_1 \xrightarrow{F} U_2 \xrightarrow{G} U_3$  be a 3-term  $C^\infty$ -chain complex such that, for each  $x_1 \in U_1$  the image of  $dF|_{x_1}$  is the entire null space of  $dG|_{F(x_1)}$ . Suppose moreover that we can find two smooth tame maps*

$$\tilde{F} : U_1 \times \mathcal{F}_2 \rightarrow \mathcal{F}_1 \quad \text{and} \quad \tilde{G} : U_1 \times \mathcal{F}_3 \rightarrow \mathcal{F}_2$$

*with both  $\tilde{F}(x_1) : \mathcal{F}_2 \rightarrow \mathcal{F}_1$  and  $\tilde{G}(x_1) : \mathcal{F}_3 \rightarrow \mathcal{F}_2$  linear maps such that*

$$(2.3) \quad dF|_{x_1} \circ \tilde{F}(x_1) + \tilde{G}(x_1) \circ dG|_{F(x_1)} = Id|_{\mathcal{F}_2}$$

*for all  $x_1 \in U_1$ . Then, for any  $x_1 \in U_1$ , the image of a neighborhood of  $x_1$  by  $F$  fills out a neighborhood of  $F(x_1)$  in the subset  $U_2 \cap \{G = 0\} \subset \mathcal{F}_2$ . Moreover we can find a smooth tame map*

$$H : U'_2 \subset \mathcal{F}_2 \longrightarrow U'_1 \subset \mathcal{F}_1$$

*from a neighborhood  $U'_2$  of  $F(x_1)$  to a neighborhood  $U'_1$  of  $x_1$  such that*

$$(F \circ H)|_{U'_2 \cap \{G=0\}} = Id|_{U'_2 \cap \{G=0\}}.$$

We close this subsection pointing out that the tangent space condition (2.3) can be viewed as a smooth family (parametrized by  $x_1 \in U_1$ ) of chain homotopy exhibiting the identity as null-homotopic,

$$\begin{array}{ccccc} \mathcal{F}_1 & \xrightarrow{dF|_{x_1}} & \mathcal{F}_2 & \xrightarrow{dG|_{F(x_1)}} & \mathcal{F}_3 \\ \downarrow Id & \swarrow \tilde{F}(x_1) & \downarrow Id & \swarrow \tilde{G}(x_1) & \downarrow Id \\ \mathcal{F}_1 & \xrightarrow{dF|_{x_1}} & \mathcal{F}_2 & \xrightarrow{dG|_{F(x_1)}} & \mathcal{F}_2 \end{array}$$

and in particular

$$\mathcal{F}_1 \xrightarrow{dF|_{x_1}} \mathcal{F}_2 \xrightarrow{dG|_{F(x_1)}} \mathcal{F}_3$$

is exact for all  $x_1 \in U_1$  in a tame and smooth way.

**2.2. A finite dimensional version of the Nash-Moser Theorem.** In this subsection we give a detailed proof of a finite dimensional version of the Nash-Moser theorem for exact sequences due to Hamilton.

We begin by recalling some basic facts. Let  $X$  be a topological space and let  $f : X \rightarrow \mathbb{R}$ . Recall that  $f$  is said to be *lower semi-continuous* at  $x_0 \in X$  if for every  $\epsilon > 0$  there exists a neighborhood  $U$  of  $x_0$  such that  $f(x_0) - \epsilon \leq f(x)$  for every  $x \in U$ . This is equivalent to say that

$$\liminf_{x \rightarrow x_0} f(x) \geq f(x_0).$$

In particular, if  $X$  is a topological space and  $A : X \rightarrow L(\mathbb{R}^m, \mathbb{R}^n)$  is a continuous map, it is easy to see that the function  $f(x) = \text{rank}(A(x))$  is lower semi-continuous.

Finally, we recall the well known Constant Rank Theorem.

**Theorem 2.2.** *Let  $U \subset \mathbb{R}^n$  be open and  $a \in U$ . Let  $f : U \rightarrow \mathbb{R}^k$  be a  $C^\infty$  function of constant rank  $r$  in  $U$ , that is  $\text{rank}(df|_x) = r$  for  $x \in U$ . Then, there are open sets  $U_1 \subset U$ ,  $U_2 \subset \mathbb{R}^n$  and  $U_3 \subset \mathbb{R}^k$ , and diffeomorphisms  $\phi : U_1 \rightarrow U_2$  and  $\psi : U_3 \rightarrow U_3$ , such that  $a \in U_1$  and  $\tilde{f}(x_1, \dots, x_n) =$*

$(\psi \circ f \circ \phi^{-1})(x_1, \dots, x_n) = (x_1, \dots, x_r, 0, \dots, 0)$  for all  $(x_1, \dots, x_n) \in U_2$ . That is, the following diagram commute,

$$\begin{array}{ccc} U_1 & \xrightarrow{f} & U_3 \\ \downarrow \phi & & \downarrow \psi \\ U_2 & \xrightarrow{\tilde{f}} & U_3 \end{array}$$

We now state and prove a finite dimensional version of the Nash-Moser theorem for exact sequences of Hamilton.

**Theorem 2.3.** *Let  $U \subset \mathbb{R}^m$  and  $V \subset \mathbb{R}^n$  be open sets and let*

$$(2.4) \quad U \xrightarrow{F} V \xrightarrow{G} \mathbb{R}^k$$

*be a 3-term  $C^\infty$ -chain complex such that  $G(F(x)) = 0$  for every  $x \in U$ . Fix  $a \in U$  and let  $b = F(a) \in V$ . If the sequence*

$$(2.5) \quad \mathbb{R}^m \xrightarrow{dF|_a} \mathbb{R}^n \xrightarrow{dG|_b} \mathbb{R}^k$$

*is exact, there is an open neighborhood  $W \subset V \subset \mathbb{R}^n$  of  $b$  and a  $C^\infty$  map  $H : W \rightarrow U \subset \mathbb{R}^m$  such that  $F(H(x)) = x$ , for every  $x \in W \cap \{G = 0\}$ . In other words, the exactness of (2.5) implies the local splitting of (2.4).*

*Remark 2.4.* We point out that condition (2.5) involves only the point  $a$  instead of a whole neighborhood of  $a$ , as it does (2.3). It is also clear that the above theorem is true for functions between differentiable manifolds.

*Proof.* For any  $C^\infty$  function  $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$  the function  $x \rightarrow \text{rank}(df|_x)$  is lower semi-continuous. Hence, applying this fact to the functions  $F$  and  $G$  we obtain open sets  $\tilde{U} \subset U$  and  $\tilde{V} \subset V$  such that  $a \in \tilde{U}$ ,  $b \in \tilde{V}$ ,  $F(\tilde{U}) \subset \tilde{V}$ , and the following inequalities hold

$$(2.6) \quad \text{rank}(dF|_a) \leq \text{rank}(dF|_x) \quad \text{for every } x \in \tilde{U},$$

$$(2.7) \quad \text{rank}(dG|_b) \leq \text{rank}(dG|_y) \quad \text{for every } y \in \tilde{V}.$$

Now, since  $G(F(x)) = 0$  for  $x \in U$ , we have  $\text{Im}(dF|_x) \subset \text{Ker}(dG|_{F(x)})$  for every  $x \in U$ . Hence,

$$(2.8) \quad \text{rank}(dG|_{F(x)}) + \text{rank}(dF|_x) \leq \text{rank}(dG|_{F(x)}) + \dim(\text{Ker}(dG|_{F(x)})) = n,$$

for every  $x \in U$ . Now, the equality in the right hand side of (2.8) is independent of the point  $F(x) \in V$ , then we have

$$(2.9) \quad \text{rank}(dG|_{F(x)}) + \text{rank}(dF|_x) \leq \text{rank}(dG|_b) + \dim(\text{Ker}(dG|_b)),$$

for every  $x \in U$ . On the other hand, since the sequence (2.5) is exact,  $\text{rank}(dF|_a) = \dim(\text{Ker}(dG|_b))$ , hence we obtain

$$(2.10) \quad \text{rank}(dG|_{F(x)}) + \text{rank}(dF|_x) \leq \text{rank}(dG|_b) + \text{rank}(dF|_a)$$

for every  $x \in U$ . Finally, combining the inequalities (2.10), (2.6) and (2.7), it follows that the equality holds in (2.10) for every  $x \in \tilde{U}$ , therefore

$$(2.11) \quad (\text{rank}(dG|_{F(x)}) - \text{rank}(dG|_b)) + (\text{rank}(dF|_x) - \text{rank}(dF|_a)) = 0$$

for every  $x \in \tilde{U}$ . In view of (2.6) and (2.7) the left hand side of (2.11) is the sum of two non-negative integers, then we obtain that

$$(2.12) \quad \text{rank}(dF|_x) = \text{rank}(dF|_a) \quad \text{for every } x \in \tilde{U}$$

and

$$(2.13) \quad \text{rank}(dG|_{F(x)}) = \text{rank}(dG|_b) \quad \text{for every } x \in \tilde{U}.$$

Set  $r = \text{rank}(dF|_a)$  and  $s = \text{rank}(dG|_b)$ , then from the exactness of the sequence (2.5) we have  $n = r + s$ .

Since  $\text{rank}(dF|_x) = r$  for  $x \in \tilde{U}$ , it follows from Theorem 2.2 (changing  $\tilde{U}$  and  $\tilde{V}$  if necessary) that there are open sets  $U_0 \subset \mathbb{R}^m$  and  $V_0 \subset \mathbb{R}^n$ , and diffeomorphisms  $\phi : \tilde{U} \rightarrow U_0$  and  $\psi : \tilde{V} \rightarrow V_0$  such that  $\phi(a) = 0 \in U_0$ ,  $\psi(b) = 0 \in V_0$ , and the function  $\tilde{F} = \psi \circ F \circ \phi^{-1} : U_0 \rightarrow V_0$  is given by

$$(2.14) \quad \tilde{F}(x_1, \dots, x_m) = (x_1, \dots, x_r, 0, \dots, 0)$$

for every  $(x_1, \dots, x_m) \in U_0$ .

Assume that  $V_0 = W_1 \times W_2 \subset \mathbb{R}^n = \mathbb{R}^r \times \mathbb{R}^s$ , where  $W_1$  and  $W_2$  are open neighborhood of zero in  $\mathbb{R}^r$  and  $\mathbb{R}^s$ , and let  $(y_1, \dots, y_r)$ ,  $(z_1, \dots, z_s)$  and  $(y_1, \dots, y_r, z_1, \dots, z_s)$  be the coordinates in  $\mathbb{R}^r$ ,  $\mathbb{R}^s$  and  $\mathbb{R}^r \times \mathbb{R}^s$  respectively. Then, in view of (2.14), for any  $(y, 0) = (y_1, \dots, y_r, 0, \dots, 0) \in W_1 \times W_2$  there exists  $(y, z) = (y_1, \dots, y_r, z_1, \dots, z_s) \in U_0$  such that  $\tilde{F}(y, z) = (y, 0)$ .

Set  $G_1 = G \circ \psi^{-1} : W_1 \times W_2 \rightarrow \mathbb{R}^k$ . Then, since  $G(F(x)) = 0$  for  $x \in \tilde{U}$ , it follows that

$$(2.15) \quad G_1(y, 0) = G(\psi^{-1}(y, 0)) = G(\psi^{-1}(\tilde{F}(y, z))) = G(F(\phi^{-1}(y, z))) = 0,$$

for every  $(y, 0) \in W_1 \times W_2$ . Hence,

$$(2.16) \quad \frac{\partial G_1}{\partial y_j}(y, 0) = 0 \quad \text{for } j = 1, \dots, r \quad \text{and } (y, 0) \in W_1 \times W_2.$$

Therefore, if  $G_1(y, w) = (g_1(y, w), \dots, g_k(y, w))$ , where  $g_i : W_1 \times W_2 \rightarrow \mathbb{R}$  are  $C^\infty$  functions for  $1 \leq i \leq k$ , we have

$$(2.17) \quad \frac{\partial g_i}{\partial y_j}(y, 0) = 0 \quad \text{for } 1 \leq i \leq k \quad \text{and } 1 \leq j \leq r,$$

and every  $(y, 0) \in W_1 \times W_2$ . Hence,

$$(2.18) \quad dG_1|_{(0,0)} = \begin{bmatrix} 0 & \cdots & 0 & \frac{\partial g_1}{\partial z_1}(0,0) & \cdots & \frac{\partial g_1}{\partial z_s}(0,0) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \frac{\partial g_k}{\partial z_1}(0,0) & \cdots & \frac{\partial g_k}{\partial z_s}(0,0) \end{bmatrix}.$$

Now, since  $d\psi^{-1}|_{(y,0)}$  is a isomorphism and  $\phi^{-1}(y, z) \in \tilde{U}$ , it follows from (2.13) that

$$(2.19) \quad \text{rank}(dG_1|_{(y,0)}) = \text{rank}(dG|_{\psi^{-1}(y,0)}) = \text{rank}(dG|_{F(\phi^{-1}(y,z))}) = s$$

for every  $(y, 0) \in W_1 \times W_2$ . In particular,

$$\text{rank}(dG_1|_{(0,0)}) = s.$$



Hence the matrix

$$(2.20) \quad \begin{bmatrix} \frac{\partial g_1}{\partial z_1}(0,0) & \cdots & \frac{\partial g_1}{\partial z_s}(0,0) \\ \vdots & \ddots & \vdots \\ \frac{\partial g_k}{\partial z_1}(0,0) & \cdots & \frac{\partial g_k}{\partial z_s}(0,0) \end{bmatrix}$$

has  $s$  linearly independent rows. If these rows correspond to the indices  $1 \leq i_1 < i_2 < \cdots < i_s \leq k$ , let  $\pi : \mathbb{R}^k \rightarrow \mathbb{R}^s$  be given by  $\pi(x_1, \dots, x_k) = (x_{i_1}, \dots, x_{i_s})$  and define

$$\tilde{G}_1 = \pi \circ G_1 : W_1 \times W_2 \subset \mathbb{R}^r \times \mathbb{R}^s \rightarrow \mathbb{R}^s.$$

Next we apply the Implicit Function Theorem to  $\tilde{G}_1$  in a neighborhood of  $(0,0) \in W_1 \times W_2$ . Since  $G(F(x)) = 0$  for every  $x \in \tilde{U}$  we have,

$$\tilde{G}_1(0,0) = \pi(G(\psi^{-1}(0,0))) = \pi(G(b)) = \pi(G(F(a))) = \pi(0) = 0.$$

Moreover, since  $\tilde{G}_1(y,z) = (g_{i_1}(y,z), \dots, g_{i_s}(y,z))$  and the rows of (2.20) corresponding to the indices  $1 \leq i_1 < i_2 < \cdots < i_s \leq k$  are linearly independent, the differential of  $\tilde{G}_1$  with respect to the variable  $z$  at  $(0,0)$  is not singular, that is, the matrix

$$\begin{bmatrix} \frac{\partial g_{i_1}}{\partial z_1}(0,0) & \cdots & \frac{\partial g_{i_1}}{\partial z_s}(0,0) \\ \vdots & \ddots & \vdots \\ \frac{\partial g_{i_s}}{\partial z_1}(0,0) & \cdots & \frac{\partial g_{i_s}}{\partial z_s}(0,0) \end{bmatrix}$$

is invertible. Then, by the Implicit Function Theorem there are open neighborhoods,  $0 \in U_1 \subset W_1$  and  $0 \in U_2 \subset W_2$ , and a unique  $C^\infty$  function  $g : U_1 \rightarrow U_2$  such that,

$$(i) \quad g(0) = 0$$

$$(ii) \quad \tilde{G}_1(y, g(y)) = 0 \quad \text{for every } y \in U_1.$$

$$(iii) \quad \text{If } y \in U_1 \text{ and } w \in U_2 \text{ are such that } \tilde{G}_1(y, w) = 0 \text{ then } g(y) = w.$$

In particular, it follows from (2.15) that  $\tilde{G}_1(y,0) = \pi(G_1(y,0)) = \pi(0) = 0$  for  $(y,0) \in U_1 \times U_2$ . Then from (iii) we have,

$$(2.21) \quad g(y) = 0 \quad \text{for every } y \in U_1.$$

Now define  $\tilde{H} : W_1 \times W_2 \subset \mathbb{R}^r \times \mathbb{R}^s \rightarrow \mathbb{R}^r \times \mathbb{R}^{m-r}$  by  $\tilde{H}(y,w) = (y,0)$ , set  $W = \psi^{-1}(U_1 \times U_2) \subset \mathbb{R}^n$  and define  $H = \phi^{-1} \circ \tilde{H} \circ \psi : W \rightarrow \mathbb{R}^m$ . Then,  $W$  is an open neighborhood of  $b \in \mathbb{R}^n$  and  $H$  is a  $C^\infty$  function.

Let  $x \in W \cap \{G = 0\}$  and let  $\psi(x) = (x_1, x_2) \in U_1 \times U_2$ . Then,  $\tilde{G}_1(x_1, x_2) = \pi(G(\psi^{-1}(x_1, x_2))) = \pi(G(x)) = \pi(0) = 0$ . Therefore, from (iii) and (2.21) it follows that  $x_2 = g(x_1) = 0$ . Hence  $\psi(x) = (x_1, 0) \in U_1 \times U_2$  for every  $x \in W \cap \{G = 0\}$ . Then we have,

$$\begin{aligned} F(H(x)) &= F(\phi^{-1} \circ \tilde{H} \circ \psi(x)) = F(\phi^{-1}(\tilde{H}(x_1, 0))) \\ &= (F \circ \phi^{-1})(x_1, 0) = (\psi^{-1} \circ \tilde{F})(x_1, 0) \\ &= \psi^{-1}(\tilde{F}(x_1, 0)) = \psi^{-1}(x_1, 0) = x, \end{aligned}$$

for every  $x \in W \cap \{G = 0\}$ . This completes the proof of the theorem.  $\square$



### 3. RIGIDITY OF LIE ALGEBRAS

#### 3.1. The Chevalley-Eilenberg complex and rigidity of Lie algebras.

In this subsection we briefly recover from Theorem 2.3, and for any Lie algebra  $\mathfrak{g}$ , the well known fact stating that if  $H^2(\mathfrak{g}, \mathfrak{g}) = 0$  then  $\mathfrak{g}$  is rigid.

Given a Lie algebra  $\mathfrak{g}$  with Lie bracket  $\mu$  and a representation  $(\pi, V)$  of  $\mathfrak{g}$ , one obtains a chain complex called the Chevalley-Eilenberg complex of  $\mathfrak{g}$  with coefficients in  $V$  as follows. For any  $k \in \mathbb{Z}_{\geq 0}$  the space of  $k$ -cochains  $C^k(\mathfrak{g}, V)$  is given by  $\bigwedge^k \mathfrak{g}^* \otimes V$ , or equivalently by the space of all  $k$ -multilinear and skew-symmetric maps  $\omega : \mathfrak{g} \times \cdots \times \mathfrak{g} \rightarrow V$ . The coboundary operator in degree  $k$  is the map  $d_\mu^k : C^k(\mathfrak{g}, V) \rightarrow C^{k+1}(\mathfrak{g}, V)$  defined by,

$$\begin{aligned} (d_\mu^k \omega)(x_0, \dots, x_k) &= \sum_{i=0}^k (-1)^i \pi(x_i)(\omega(x_0, \dots, \hat{x}_i, \dots, x_k)) \\ &+ \sum_{0 \leq i < j \leq k} (-1)^{i+j} \omega(\mu(x_i, x_j), x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_k), \end{aligned}$$

where  $\hat{x}_i$  denotes that the argument  $x_i$  is removed. Since  $d_\mu^k \circ d_\mu^{k-1} = 0$  it follows that  $\text{Im}(d_\mu^{k-1}) \subset \text{Ker}(d_\mu^k)$ . Then, the  $k$ -cohomology space of  $\mathfrak{g}$  with coefficients in  $V$  is

$$H^k(\mathfrak{g}, V) = \text{Ker}(d_\mu^k) / \text{Im}(d_\mu^{k-1}).$$

In particular, if  $V = \mathfrak{g}$  and  $\pi = \text{ad}_\mathfrak{g}$  is the adjoint representation of  $\mathfrak{g}$  the corresponding  $k$ -cohomology space will be denoted by  $H^k(\mathfrak{g}, \mathfrak{g})$ , and it is called the  $k$ -adjoint cohomology space of  $\mathfrak{g}$ .

It turns out that the following portion of the Chevalley-Eilenberg complex for the adjoint cohomology  $\mathfrak{g}$ ,

$$\mathfrak{g}^* \otimes \mathfrak{g} \xrightarrow{d_\mu^1} \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g} \xrightarrow{d_\mu^2} \Lambda^3 \mathfrak{g}^* \otimes \mathfrak{g}$$

is the sequence (2.5) for the following 3-term  $C^\infty$ -chain complex,

$$(3.1) \quad GL(\mathfrak{g}) \xrightarrow{F} \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g} \xrightarrow{G=J} \Lambda^3 \mathfrak{g}^* \otimes \mathfrak{g},$$

where

$$\begin{aligned} F(g) &= g \cdot \mu = g(\mu(g^{-1}x, g^{-1}y)); \\ J(\sigma)(x, y, z) &= \sigma(x, \sigma(y, z)) + \sigma(y, \sigma(z, x)) + \sigma(z, \sigma(x, y)), \end{aligned}$$

here  $g \in GL(\mathfrak{g})$ ,  $\sigma \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$  and  $x, y, z \in \mathfrak{g}$ . Since  $\mu$  is a Lie algebra structure and  $J$  is the Jacobi operator we have,  $J(F(g)) = J(g \cdot \mu) = 0$  for every  $g \in GL(\mathfrak{g})$ . Now, we apply Theorem 2.3 for  $a = I \in GL(\mathfrak{g})$  and  $b = \mu \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$ . Since

$$(3.2) \quad dF|_I = d_\mu^1 \quad \text{and} \quad dJ|_\mu = d_\mu^2,$$

it follows that the sequence

$$(3.3) \quad T_I(GL(\mathfrak{g})) \xrightarrow{dF|_I} T_\mu(\Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}) \xrightarrow{dJ|_\mu} T_0(\Lambda^3 \mathfrak{g}^* \otimes \mathfrak{g}),$$

is exact if and only if  $H^2(\mathfrak{g}, \mathfrak{g}) = 0$ . Therefore, if  $H^2(\mathfrak{g}, \mathfrak{g}) = 0$  then there exist an open neighborhood  $W$  of  $\mu$  in  $\Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$  and a  $C^\infty$  map  $H : W \rightarrow GL(\mathfrak{g})$

such that

$$(3.4) \quad H(\lambda) \cdot \mu = F(H(\lambda)) = \lambda,$$

for every  $\lambda \in W \cap \{J = 0\} = W \cap \mathcal{L}_n$ . In other words, every Lie algebra in a neighborhood of  $\mu$  is conjugate to  $\mu$  under  $GL(\mathfrak{g})$ , hence  $(\mathfrak{g}, \mu)$  is rigid in  $\mathcal{L}_n$ . This completes the proof of the following theorem.

**Theorem 3.1.** *Let  $(\mathfrak{g}, \mu)$  be a Lie algebra over  $\mathbb{R}$ . If  $H^2(\mathfrak{g}, \mathfrak{g}) = 0$  there is a neighborhood  $W$  of  $\mu$  in  $\Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$  and a smooth map  $H : W \rightarrow GL(\mathfrak{g})$  such that  $H(\lambda) \cdot \mu = \lambda$  for every  $\lambda \in W \cap \mathcal{L}_n$ . Hence  $(\mathfrak{g}, \mu)$  is rigid in  $\mathcal{L}_n$ .*

**3.2. Rigidity in the variety  $k$ -step nilpotent Lie algebras.** Let  $\{\mathfrak{g}^i\}_{i \geq 0}$  and  $\{\mathfrak{g}^{(i)}\}_{i \geq 0}$  denote, respectively, the descending central and derived series of a Lie algebra  $\mathfrak{g}$  ( $\mathfrak{g}^1 = \mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}]$ ). Also, let

$$\mathcal{N}_{n,k} = \{\mathfrak{g} \in \mathcal{L}_n : \mathfrak{g}^k = 0\} \quad \text{and} \quad \mathcal{S}_{n,k} = \{\mathfrak{g} \in \mathcal{L}_n : \mathfrak{g}^{(k)} = 0\}$$

be, respectively, the subvariety of all (at most)  $k$ -step nilpotent and solvable Lie algebras. Now, we will use Theorem 2.3 to discuss rigidity in  $\mathcal{N}_{n,k}$  and  $\mathcal{S}_{n,k}$ .

Let  $\mathfrak{g}$  be a vector space over  $\mathbb{R}$  of dimension  $n$ . For  $k \geq 1$  consider the maps

$$(3.5) \quad N_k \in \text{Hom}(\Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}, (\mathfrak{g}^*)^{\otimes(k+1)} \otimes \mathfrak{g}),$$

defined inductively as follows,

$$N_1(\mu)(x_1, x_2) = \mu(x_1, x_2),$$

$$N_k(\mu)(x_1, \dots, x_k, x_{k+1}) = \mu(N_{k-1}(\mu)(x_1, \dots, x_k), x_{k+1}),$$

where  $\mu \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$ . It is clear that

$$(3.6) \quad \mathcal{N}_{n,k} = \{\mu \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g} : J(\mu) = 0 \text{ and } N_k(\mu) = 0\}.$$

It is not difficult to see that  $dN_k|_\mu : \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g} \rightarrow (\mathfrak{g}^*)^{\otimes(k+1)} \otimes \mathfrak{g}$  is given by,

$$dN_1|_\mu(\sigma)(x_1, x_2) = \sigma(x_1, x_2),$$

$$dN_2|_\mu(\sigma)(x_1, x_2, x_3) = \mu(\sigma(x_1, x_2), x_3) + \sigma(\mu(x_1, x_2), x_3),$$

$$dN_3|_\mu(\sigma)(x_1, x_2, x_3, x_4) = \mu(\mu(\sigma(x_1, x_2), x_3), x_4) + \mu(\sigma(\mu(x_1, x_2), x_3), x_4) + \sigma(\mu(\mu(x_1, x_2), x_3), x_4),$$

and so on for  $k \geq 4$ .

Since  $\text{Im}(d_\mu^1) \subset \text{Ker}(d_\mu^2) \cap \text{Ker}(dN_k|_\mu)$ , in analogy with the definition of the second cohomology space  $H^2(\mathfrak{g}, \mathfrak{g})$  we define

$$(3.7) \quad H_{k\text{-nil}}^2(\mathfrak{g}, \mathfrak{g}) = \frac{\text{Ker}(d_\mu^2) \cap \text{Ker}(dN_k|_\mu)}{\text{Im}(d_\mu^1)}$$

for any  $k$ -step nilpotent Lie algebra  $(\mathfrak{g}, \mu)$ . In the following theorem we apply Theorem 2.3 to obtain a rigidity result similar to Theorem 3.1 for  $k$ -step nilpotent Lie algebras.

**Theorem 3.2.** *Let  $(\mathfrak{g}, \mu)$  be a  $k$ -step nilpotent Lie algebra over  $\mathbb{R}$ . Then if  $H_{k\text{-nil}}^2(\mathfrak{g}, \mathfrak{g}) = 0$  there is a neighborhood  $W$  of  $\mu$  in  $\Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$  and a smooth map  $H : W \rightarrow GL(\mathfrak{g})$  such that  $H(\lambda) \cdot \mu = \lambda$  for every  $\lambda \in W \cap \mathcal{N}_{n,k}$ . Hence  $(\mathfrak{g}, \mu)$  is rigid in  $\mathcal{N}_{n,k}$ .*

*Proof.* Let  $(\mathfrak{g}, \mu)$  be a  $k$ -step nilpotent Lie algebra over  $\mathbb{R}$  and let  $F$  and  $J$  be as in (3.1). If  $N_k$  is the function defined above, we consider the following 3-term  $C^\infty$ -chain complex

$$(3.8) \quad GL(\mathfrak{g}) \xrightarrow{F} \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g} \xrightarrow{G=J \oplus N_k} \left( \Lambda^3 \mathfrak{g}^* \otimes \mathfrak{g} \right) \oplus \left( (\mathfrak{g}^*)^{\otimes(k+1)} \otimes \mathfrak{g} \right).$$

Since  $\mu$  is a  $k$ -step nilpotent Lie algebra we have  $G(F(g)) = 0$  for  $g \in GL(\mathfrak{g})$ . Next we apply Theorem 2.3 for  $a = I \in GL(\mathfrak{g})$  and  $b = \mu \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$ . Since

$$dF|_I = d_\mu^1 \quad \text{and} \quad dG|_\mu = d_\mu^2 \oplus dN_k|_\mu,$$

it follows that the sequence

$$(3.9) \quad \mathfrak{g}^* \otimes \mathfrak{g} \xrightarrow{dF|_I} T_\mu(\Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}) \xrightarrow{dG|_\mu} T_0\left(\Lambda^3 \mathfrak{g}^* \otimes \mathfrak{g}\right) \oplus T_0\left((\mathfrak{g}^*)^{\otimes(k+1)} \otimes \mathfrak{g}\right),$$

is exact if and only if  $H_{k-nil}^2(\mathfrak{g}, \mathfrak{g}) = 0$ . Hence, from Theorem 2.3, it follows that if  $H_{k-nil}^2(\mathfrak{g}, \mathfrak{g}) = 0$  there exist a neighborhood  $W$  of  $\mu$  in  $\Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$  and a  $C^\infty$  function  $H : W \rightarrow GL(\mathfrak{g})$  such that

$$(3.10) \quad H(\lambda) \cdot \mu = F(H(\lambda)) = \lambda,$$

for every  $\lambda \in W \cap \{J \oplus N_k = 0\} = W \cap \mathcal{N}_{n,k}$  (see (3.6)). Hence  $(\mathfrak{g}, \mu)$  is rigid in  $\mathcal{N}_{n,k}$ , as we wanted to proof.  $\square$

**3.3. The radical of the polynomial ideal defining  $\mathcal{N}_{n,k}$ .** Let  $\mathfrak{g}$  be an  $n$ -dimensional vector space over  $\mathbb{R}$ . We know that  $J(\mu)$  and  $N_k(\mu)$  (for  $\mu \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$ ), when expressed in coordinates, are polynomials of degree 2 and  $k$ , respectively. It turns out that some algebraic properties of the ideal generated by these polynomials depend strongly on  $n$  as we will show briefly in this subsection. These results are needed in §4.3.

Let us consider, for  $k \geq 3$ , the polynomial in  $\mu \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$  given by,

$$(3.11) \quad SN_k(\mu)(x_1, \dots, x_{k+1}) = \mu(\mu(x_1, x_2), N_{k-2}(\mu)(x_3, \dots, x_{k+1})),$$

and define

$$\mathcal{SN}_{n,k} = \{\mu \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g} : J(\mu) = 0 \text{ and } SN_k(\mu) = 0\} \subset \mathcal{L}_n.$$

It is clear that if  $\mu \in \mathcal{SN}_{n,k}$ , then  $\mu$  defines a solvable Lie algebra. More precisely,

$$(3.12) \quad \mathcal{N}_{n,k} \subset \mathcal{SN}_{n,k} \subset \mathcal{S}_{n, \lceil \log_2(k-1) \rceil + 1} \subset \mathcal{S}_{n,k-1} \subset \mathcal{L}_n.$$

This follows since  $\mathfrak{g}^{(i)} \subset \mathfrak{g}^{2^i-1}$  ( $i \in \mathbb{N}$ ) and  $N_k(\mu) = J(\mu) = 0$  imply  $SN_k(\mu) = 0$ .

The main goal of this subsection is to point out that

$$(3.13) \quad \text{in general, } \mathcal{N}_{n,k+1} \not\subset \mathcal{SN}_{n,k} \quad \text{for } k < n-1,$$

$$(3.14) \quad \text{but, } \mathcal{N}_{n,k+1} \subset \mathcal{SN}_{n,k} \quad \text{for certain } k < n-1,$$

(note that  $\mathcal{N}_{n,k} = \mathcal{N}_{n,n-1} \subset \mathcal{SN}_{n,n-1}$  for  $k \geq n$ ). The inclusion (3.14) is remarkable to us since it reveals some instances where the ideal generated by  $J(\mu)$  and  $N_k(\mu)$  is not radical with  $k < n$ .

We begin with two examples showing (3.13). Let  $\mathfrak{g}$  be the Lie algebra denoted by 12346<sub>E</sub> in [Se1] and by  $\mathfrak{g}_{6,14}$  in [CdGS]. The structure table of  $\mathfrak{g}$  is the following

$$\mathfrak{g} : [a, b] = c, [a, c] = d, [a, d] = e, [b, c] = e, [b, e] = f, [c, d] = -f.$$

It turns out that  $\mathfrak{g} \in \mathcal{N}_{6,5}$  (in fact  $\mathfrak{g}$  is rigid in  $\mathcal{N}_{6,5}$ , see §4.2), but  $\mathfrak{g} \notin \mathcal{SN}_{6,4}$ .

A more general example is the following. Let  $\mathfrak{h}_m$  be the Heisenberg Lie algebra  $[x_i, y_i] = z$  ( $i = 1, \dots, m$ ) and let  $D \in \text{Der}(\mathfrak{h}_m)$  be the derivation defined by

$$\begin{aligned} D(x_i) &= x_{i+1}, & i &= 1, \dots, m-1; \\ D(y_i) &= -y_{i-1}, & i &= 2, \dots, m. \end{aligned}$$

Then  $\mathfrak{n} = \mathbb{R}D \ltimes \mathfrak{h}_m$  is an  $(m+1)$ -step nilpotent Lie algebra of dimension  $2m+2$ , hence  $\mathfrak{n} \in \mathcal{N}_{2m+2, m+1}$ . However, it is easy to verify that  $\mathfrak{n} \notin \mathcal{SN}_{2m+2, m}$ . In particular, this shows that

$$\begin{aligned} \mathcal{N}_{n,4} &\not\subset \mathcal{SN}_{n,3} & \text{for } n = 8 \text{ and} \\ \mathcal{N}_{n,6} &\not\subset \mathcal{SN}_{n,5} & \text{for } n = 12. \end{aligned}$$

On the other hand, we have checked, using the classification, that

$$\mathcal{N}_{n,4} \subset \mathcal{SN}_{n,3} \quad \text{and} \quad \mathcal{N}_{n,6} \subset \mathcal{SN}_{n,5} \quad \text{for } n \leq 7$$

(recall that we have shown above that  $\mathcal{N}_{6,5} \not\subset \mathcal{SN}_{6,4}$ ). This shows (taking coordinates) that, for  $n \leq 7$ ,  $SN_3(\mu)$  produces polynomials of degree 3 in the radical of the ideal generated by  $J(\mu)$  and  $N_4(\mu)$ ; and  $SN_5(\mu)$  produces polynomials of degree 5 in the radical of the ideal generated by  $J(\mu)$  and  $N_6(\mu)$ . This give rise to the following question:

**Question.** Given  $k < n$ , find polynomials of degree less than  $k$  in  $\sqrt{I}$ , not in  $I$ , where  $I$  is the ideal generated by  $J(\mu)$  and  $N_k(\mu)$  in coordinates,  $\mu \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$ ,  $\dim \mathfrak{g} = n$ .

If  $P(\mu)$  is such a polynomial, it can be used as (part of) the function  $G$  in Theorem 2.3. This could help to recognize more easily rigid Lie algebras in  $\mathcal{N}_{n,k}$ . This tool is used in §4.3 to find a rigid curve in  $\mathcal{N}_{7,6}$ .

Summarizing, we have shown in this subsection that

- (i) if  $k = 4$ , then  $SN_3(\mu)$  is an example for  $n = 5, 6, 7$  but not for  $n = 8$ ;
- (ii) if  $k = 5$ , we do not know any example;
- (iii) if  $k = 6$ , then  $SN_5(\mu)$  is an example for  $n = 7$  (fails for  $n = 12$ ).

#### 4. DEFORMATIONS AND RIGIDITY IN $\mathcal{N}_{n,k}$ , FOR $n = 5, 6, 7$

In this section we will consider several structure tables of Lie algebras. In order to shorten the description of these tables we will denote by  $ab$  the Lie bracket  $[a, b]$ .

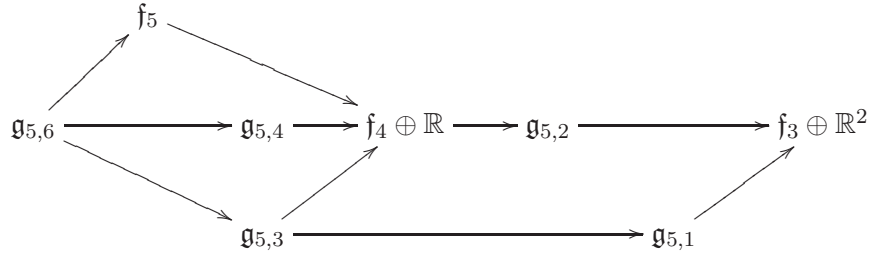
**4.1. A rigid Lie algebra with  $H_{k-nil}^2(\mathfrak{g}, \mathfrak{g}) \neq 0$ .** It is well known that a Lie algebra  $\mathfrak{g}$  may be rigid but fail to satisfy  $H^2(\mathfrak{g}, \mathfrak{g}) = 0$ ; the examples are in general involved, see for instance [R]. In this subsection we present Lie algebra  $\mathfrak{g}$  that is rigid in the variety  $\mathcal{N}_{5,3}$  but  $H_{3-nil}^2(\mathfrak{g}, \mathfrak{g}) \neq 0$ .

There are only eight non-abelian Lie algebras of dimension 5 over the real numbers [dG]. If  $\mathfrak{f}_n$  denotes the standard filiform Lie algebra of dimension  $n$ , these Lie algebras are

$k$	$k$ -step Lie algebra $\mathfrak{g}$	$\dim H_{k-nil}^2(\mathfrak{g}, \mathfrak{g})$
2	$\mathfrak{f}_3 \oplus \mathbb{R}^2$	$11 = 20 - 9$
	$\mathfrak{g}_{5,1} : ab = e, cd = e.$	$0 = 10 - 10$
	$\mathfrak{g}_{5,2} : ab = d, ac = e.$	$0 = 12 - 12$
3	$\mathfrak{f}_4 \oplus \mathbb{R}$	$4 = 18 - 14$
	$\mathfrak{g}_{5,3} : ab = d, ad = e, bc = e.$	$2 = 17 - 15$
	$\mathfrak{g}_{5,4} : ab = c, ac = d, bc = e.$	$0 = 15 - 15$
4	$\mathfrak{f}_5$	$1 = 17 - 16$
	$\mathfrak{g}_{5,6} : ab = c, ac = d, ad = e, bc = e.$	$0 = 17 - 17$

The list of 5-dimensional non-abelian nilpotent Lie algebras

and the Hasse diagram is [GO]



This shows that  $\mathfrak{g}_{5,3}$  is rigid in the variety of 3-step nilpotent Lie algebras but it turns out that  $H_{3-nil}^2(\mathfrak{g}_{5,3}, \mathfrak{g}_{5,3}) = \text{span}\{\nu_1, \nu_2\}$  with

$$\begin{aligned} \nu_1(b, c) &= c, \\ \nu_2(a, b) &= b, \quad \nu_2(a, c) = -c, \quad \nu_2(a, d) = -d. \end{aligned}$$

In fact, if  $\mu$  is the bracket of  $\mathfrak{g}_{5,3}$ , then  $\mu + t\nu_1$  and  $\mu + t\nu_2$  are solvable deformations of  $\mathfrak{g}_{5,3}$ .

**4.2. Rigid nilpotent Lie algebras in  $\mathcal{N}_{6,k}$ .** The Hasse diagram of the 6-dimensional nilpotent Lie algebras is given in [Se1]. There are 34 real 6-dimensional nilpotent Lie algebras [CdGS] and being this a finite number it follows, as in dimension 5, that a Lie algebra is rigid in its class if and only if it is not a degeneration of any other in its class. It follows the Hasse diagram in [Se1] that there are only one rigid Lie algebra in  $\mathcal{N}_{6,5}$ , three (two over  $\mathbb{C}$ ) rigid Lie algebras in  $\mathcal{N}_{6,4}$ , four (two over  $\mathbb{C}$ ) in  $\mathcal{N}_{6,3}$  and three (two over  $\mathbb{C}$ ) in  $\mathcal{N}_{6,2}$ . The following table summarizes this information.

$k$	$k$ -step Lie algebra $\mathfrak{g}$ as denoted in [Se1]	$k$ -step Lie algebra $\mathfrak{g}$ as denoted in [CdGS]	$\dim H_{k-nil}^2(\mathfrak{g}, \mathfrak{g})$
2	36	$\mathfrak{g}_{6,26}$	$0 = 18 - 18$
	$13 + 13$	$\mathfrak{g}_{6,22}, t = -1, 1$	$0 = 20 - 20$
3	$246_E$	$\mathfrak{g}_{6,24}, t = -1, 1$	$2 = 26 - 24$
	$136_A$	$\mathfrak{g}_{6,19}, t = -1, 1$	$0 = 25 - 25$
4	1246	$\mathfrak{g}_{6,13},$	$1 = 27 - 26$
	$1346_C$	$\mathfrak{g}_{6,21}, t = -1, 1$	$0 = 26 - 26$
5	$12346_E$	$\mathfrak{g}_{6,14}$	$0 = 28 - 28$

The list of rigid 6-dimensional nilpotent Lie algebras in  $\mathcal{N}_{6,k}$

In this case there are three rigid nilpotent Lie algebras with non-zero cohomology  $H_{k-nil}^2(\mathfrak{g}, \mathfrak{g})$ , these are  $\mathfrak{g}_{6,13}$  and  $\mathfrak{g}_{6,24}, t = -1, 1$ ; and the non-zero cohomology classes correspond to infinitesimal solvable deformations.

**4.3. The two rigid curves in  $\mathcal{N}_7$ .** It is known that there are no rigid Lie algebras in  $\mathcal{N}_7$  (see [Ca]) and that there are only three (two over  $\mathbb{C}$ ) rigid curves of non-isomorphic Lie algebras in  $\mathcal{N}_7$  (see [GA]).

One of these curves consists of 6-step nilpotent Lie algebras and it is denoted as  $\mathfrak{g}_I(\alpha)$  by Burde [Bu], as  $\mathfrak{g}_{7,1.1}(ii_\lambda)$  by Magnin [Ma] and as  $123457_I$  by Seeley [Se2]. If  $\mathfrak{g}_6(r, t)$  is the surface of (solvable) Lie algebras given by the following structure table

$$\begin{aligned} \mathfrak{g}_6(r, t) : \quad & ab = c, \quad ac = d, \quad ad = e, \quad ae = f, \quad af = g, \quad ag = rg, \\ & bc = e, \quad bd = f, \quad be = rtf + (1 - t)g, \quad bf = rg, \quad bg = r^2g, \\ & \quad \quad \quad cd = -rtf + tg, \end{aligned}$$

then  $\mathfrak{g}_6(0, \alpha)$  is exactly  $\mathfrak{g}_I(\alpha)$ .

The other two curves consist of 5-step nilpotent Lie algebras and they coincide over  $\mathbb{C}$ . Over the complex numbers, this curve is denoted as  $\mathfrak{g}_1(\lambda)$  by Burde [Bu], as  $\mathfrak{g}_{7,0.4}(\lambda)$  by Magnin [Ma] and as  $12457_N$  by Seeley [Se2]. The structure table of  $\mathfrak{g}_1(\lambda)$  is obtained by setting  $(r, t) = (0, \lambda)$  in the following surface of solvable Lie algebras,

$$\begin{aligned} \mathfrak{g}_5(r, t) : \quad & ab = (1 + tr)c, \quad ac = d, \quad ad = f + tg, \quad ae = g, \quad af = -rf + g, \\ & bc = e, \quad bd = g, \quad be = rd + f, \quad ce = g. \end{aligned}$$

It is easy to check that both surfaces,  $\{\mathfrak{g}_5(r, t)\}_{(r,t) \in \mathbb{R}^2}$  and  $\{\mathfrak{g}_6(r, t)\}_{(r,t) \in \mathbb{R}^2}$ , are contained in  $\mathcal{SN}_{7,5}$  (see §3.3 for the definition). In addition, a Lie algebra in either of these curves is nilpotent if and only if  $r = 0$ .

Therefore if  $\mu(r, t)$  is the Lie algebra structure of either  $\mathfrak{g}_5(r, t)$  or  $\mathfrak{g}_6(r, t)$  we have the following 3-term  $C^\infty$ -chain complex

$$\mathbb{R}^2 \times GL(\mathfrak{g}) \xrightarrow{F} \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g} \xrightarrow{G=J \oplus SN_5} \left( \Lambda^3 \mathfrak{g}^* \otimes \mathfrak{g} \right) \oplus \left( (\mathfrak{g}^*)^{\otimes 6} \otimes \mathfrak{g} \right),$$

where  $SN_5$  is as in (3.11) and  $F(r, t, g) = g \cdot \mu(r, t)$ .

The corresponding linear chain complex of the tangent spaces at the points  $(r_0, t_0, I) \in \mathbb{R}^2 \times GL(\mathfrak{g})$  and  $\mu(r_0, t_0) \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$  is

$$(4.1) \quad \mathbb{R}^2 \times \mathfrak{g}^* \otimes \mathfrak{g} \xrightarrow{dF|_{(r_0, t_0, I)}} \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g} \xrightarrow{dG|_{\mu(r_0, t_0)}} \left( \Lambda^3 \mathfrak{g}^* \otimes \mathfrak{g} \right) \oplus \left( (\mathfrak{g}^*)^{\otimes 6} \otimes \mathfrak{g} \right)$$

where

$$dF|_{(r_0, t_0, I)} = [\partial_r|_{(r_0, t_0)} \mu, \partial_t|_{(r_0, t_0)} \mu, d_{\mu(r_0, t_0)}^1],$$

$$dG|_{\mu(r_0, t_0)} = \begin{bmatrix} d_{\mu(r_0, t_0)}^2 \\ d(SN_5)|_{\mu(r_0, t_0)} \end{bmatrix}.$$

A computer calculation shows that the chain complex (4.1) is exact for all  $(r_0, t_0)$  with  $r_0 t_0 \neq -1$ , if  $\mu(r_0, t_0)$  is the structure of  $\mathfrak{g}_5(r_0, t_0)$ ; and for all  $(r_0, t_0)$  with  $t_0 \neq 0$ , if  $\mu(r_0, t_0)$  is the structure of  $\mathfrak{g}_6(r_0, t_0)$ .

**Proposition 4.1.** *The curves  $\{\mathfrak{g}_1(\lambda) : \lambda \in \mathbb{R}\}$  and  $\{\mathfrak{g}_I(\alpha) : \alpha \in \mathbb{R}, \alpha \neq 0\}$  are rigid curves in  $\mathcal{N}_7$ . Moreover, for any  $r_0 \neq 0$ , the curves  $\{\mathfrak{g}_5(r_0, t) : t \in \mathbb{R}, t \neq -1/r_0\}$  and  $\{\mathfrak{g}_6(r_0, t) : t \in \mathbb{R}, t \neq 0\}$  are rigid curves in  $\mathcal{SN}_{7,5}$ .*

*Proof.* Recall that  $\mathfrak{g}_6(0, t) \simeq \mathfrak{g}_I(t)$  and  $\mathfrak{g}_5(0, t) \simeq \mathfrak{g}_1(t)$ . Fix  $t_0 \in \mathbb{R}$  (and  $t_0 \neq 0$  if  $\mu(0, t_0)$  is the structure of  $\mathfrak{g}_6(0, t_0)$ ). It follows from Theorem 2.3 and the exactness of (4.1) that there is a neighborhood  $U \subset \mathcal{SN}_{7,5}$  of  $\mu(0, t_0)$  such that for any Lie algebra structure  $\nu \in U$  there exists  $(r, t) \in \mathbb{R}^2$  such that  $\nu \simeq \mu(r, t)$ . If in addition  $\nu$  is nilpotent (that is  $\nu \in U \cap \mathcal{N}_7$ ), then  $r$  must be 0 as  $\mu(r, t)$  is nilpotent if and only if  $r = 0$ . This proves that  $\{\mathfrak{g}_I(\alpha) : \alpha \in \mathbb{R}, \alpha \neq 0\}$  and  $\{\mathfrak{g}_1(\lambda) : \lambda \in \mathbb{R}\}$  are rigid curves in  $\mathcal{N}_7$ .

Now fix  $r_0 \neq 0$ , and  $t_0 \neq -1/r_0$  if  $\mu(r_0, t_0)$  is the structure of  $\mathfrak{g}_5(r_0, t_0)$ ; or  $t_0 \neq 0$  if  $\mu(r_0, t_0)$  is the structure of  $\mathfrak{g}_6(r_0, t_0)$ . It turns out that a computer calculation shows that (4.1) is still exact if we consider the function  $F$  (and its differential) with the variable  $r$  fixed at  $r = r_0$ . Now Theorem 2.3 implies that there is a neighborhood  $U \subset \mathcal{SN}_{7,5}$  of  $\mu(r_0, t_0)$  such that for any Lie algebra structure  $\nu \in U$  there exists  $t \in \mathbb{R}$  such that  $\nu \simeq \mu(r_0, t)$ . This proves that  $\{\mathfrak{g}_5(r_0, t) : t \in \mathbb{R}, t \neq -1/r_0\}$  and  $\{\mathfrak{g}_6(r_0, t) : t \in \mathbb{R}, t \neq 0\}$  are rigid curves in  $\mathcal{SN}_{7,5}$ .  $\square$

*Remark 4.2.* We point out that if  $r_1 \neq r_2$  then there might be isomorphic Lie algebras in the curves  $\{\mu(r_1, t) : t \in \mathbb{R}\}$  and  $\{\mu(r_2, t) : t \in \mathbb{R}\}$ .

**4.4. Deformations and rigidity in  $\mathcal{N}_{7,3}$ .** In this subsection we will follow the classification of the 7-dimensional nilpotent Lie algebras over  $\mathbb{R}$  given by Gong in [Go] and the one given by Seeley in [Se2]. The classification of Gong corrects some errors in the list given by Seeley. A more recent classification is given by Magnin in [Ma] (see also [Ca]) but we will follow the classification of [Go] and [Se2] since these authors list the Lie algebras by their upper central series (in [Ma] the Lie algebras are listed by rank). The list of 3-step nilpotent Lie algebras of dimension 7 in [Go] has 52 isolated real Lie algebras<sup>1</sup> and two 1-parameter families of pairwise non-isomorphic nilpotent Lie algebras.

<sup>1</sup> We found that two of the isolated Lie algebras of Gong's list are respectively isomorphic to two other Lie algebras of the same list. On the other hand, we found a Lie algebra that apparently is not in Gong's list (see items 3 and 6 below). This would yield a list of 51 isolated 3-step nilpotent Lie algebras of dimension 7.



A computer calculation shows that there are three 3-step nilpotent Lie algebras  $\mathfrak{g}$  with  $H_{3-nil}^2(\mathfrak{g}, \mathfrak{g}) = 0$  and thus they are rigid in  $\mathcal{N}_{7,3}$ . These are

$$\mathfrak{g}_{247H}, \quad \mathfrak{g}_{137B}, \quad \mathfrak{g}_{137B_1}$$

( $\mathfrak{g}_{137B} \simeq \mathfrak{g}_{137B_1}$  over  $\mathbb{C}$  [Go]) and the dimension of their orbits are, respectively, 38, 36 and 36.

In addition, the two 1-parameter families mentioned above are

$$\mathfrak{g}_{147E}(t) \quad \text{and} \quad \mathfrak{g}_{147E_1}(t), \quad \text{with } t > 1$$

(over  $\mathbb{C}$ , if  $t = \cosh(\theta) > 1$  then  $\mathfrak{g}_{147E_1}(t)$  is isomorphic to  $\mathfrak{g}_{147E}(t')$  with  $t' = -\frac{(1-i \sinh(\theta))^2}{\cosh^2(\theta)} \in \mathbb{C}$ ). It turns out that, if

$$\mathfrak{g}(t) \text{ is either } \begin{cases} \mathfrak{g}_{147E_1}(t) & \text{with } t > 1, \text{ or} \\ \mathfrak{g}_{147E}(t) & \text{with } t > 1, t \neq 2, \end{cases}$$

then  $\dim H_{3-nil}^2(\mathfrak{g}(t), \mathfrak{g}(t)) = 1$  ( $\dim H_{3-nil}^2(\mathfrak{g}_{147E}(2), \mathfrak{g}_{147E}(2)) = 3$ ) and the non-zero cohomology class corresponds to the tangent vector of  $\mathfrak{g}(t)$ . Therefore, the same argument given in the proof of Proposition 4.1, proves that the curves

$$\{\mathfrak{g}_{147E}(t) : 1 < t < 2\} \quad \{\mathfrak{g}_{147E}(t) : 2 < t\} \quad \{\mathfrak{g}_{147E_1}(t) : 1 < t\}$$

are rigid in  $\mathcal{N}_{7,3}$ .

There are seven other 3-step nilpotent Lie algebras  $\mathfrak{g}$  of dimension 7 in [Go] (not members of the previous curves) such that  $\dim H_{3-nil}^2(\mathfrak{g}, \mathfrak{g}) = 1$ . These are

$$\mathfrak{g}_{247G}, \quad \mathfrak{g}_{247K}, \quad \mathfrak{g}_{147D}, \quad \mathfrak{g}_{137A}, \quad \mathfrak{g}_{137D}, \quad \mathfrak{g}_{247H_1}, \quad \mathfrak{g}_{137A_1}$$

and none of them is rigid. In fact we claim

$$\begin{array}{ll} \mathfrak{g}_{247H} \rightarrow \mathfrak{g}_{247K} & \text{(see item 5)} \\ \searrow \mathfrak{g}_{247G} \simeq \mathfrak{g}_{247H_1} & \text{(see items 3 and 4)} \\ \mathfrak{g}_{137B} \rightarrow \mathfrak{g}_{137A} & \text{(see item 1)} \\ \searrow \mathfrak{g}_{137D} & \text{(see item 1)} \\ \mathfrak{g}_{137B_1} \rightarrow \mathfrak{g}_{137A_1} & \text{(see item 1)} \\ \mathfrak{g}_{147E_1}(t) \rightarrow \mathfrak{g}_{147D}, \text{ as } t \rightarrow 1. & \text{(see item 2)} \end{array}$$

Next we give the details about this and we point out some errors (in relation to this) that apparently appear in other papers.

1. It is not difficult to see that  $\mathfrak{g}_{137B} \rightarrow \mathfrak{g}_{137A}$  and  $\mathfrak{g}_{137B_1} \rightarrow \mathfrak{g}_{137A_1}$  since

$$\begin{array}{l} \mathfrak{g}_{137A} : ab = e, ae = g, cd = f, cf = g; \\ \mathfrak{g}_{137B} : ab = e, ae = g, cd = f, cf = g, bd = g; \\ \mathfrak{g}_{137A_1} : ac = e, ad = f, ae = g, bc = -f, bd = e, bf = g; \\ \mathfrak{g}_{137B_1} : ac = e, ad = f, ae = g, bc = -f, bd = e, bf = g, cd = g. \end{array}$$

In addition, let

$$\mathfrak{g}(t) : ab = e, ad = f, af = g, bc = f, bd = g, cd = -t^2e, ce = -g.$$

If we rewrite the table of  $\mathfrak{g}(t)$  in the basis

$$\{ta + c, 2t(tb - d), -ta + c, -2t(tb + d), 4t^2(te - f), 4t^2(te + f), -8t^3g\}$$

we obtain the structure table of  $\mathfrak{g}_{137B}$ . Since  $\mathfrak{g}(0) \simeq \mathfrak{g}_{137D}$  (same structure table), it follows that  $\mathfrak{g}_{137B} \rightarrow \mathfrak{g}_{137D}$ .

2. The structure table of  $\mathfrak{g}_{147E_1}(t)$  is

$$\begin{aligned} \mathfrak{g}_{147E_1}(t) : ab = d, ac = -f, af = -tg, \\ bc = e, be = tg, bf = 2g, cd = -2g. \end{aligned}$$

If  $t = 1$  and we rewrite this table in the basis

$$\{-a, a + b, c, -d, e - f, -f, -g\}$$

we obtain the structure table of  $\mathfrak{g}_{147D}$

$$\mathfrak{g}_{147D} : ab = d, ac = -f, ae = g, af = g, bc = e, bf = g, cd = -2g.$$

3.  $\mathfrak{g}_{247H_1} \simeq \mathfrak{g}_{247G}$  over  $\mathbb{R}$  in contrast to what is claimed in [Go], page 111 and in the list of page 115. Indeed,  $\mathfrak{g}_{247H_1} \simeq \mathfrak{g}_{247G}$  over  $\mathbb{R}$  since

$$\mathfrak{g}_{247G} : ab = d, ac = e, ad = f, ae = f, bd = f, be = g, cd = g, ce = f,$$

and the change of basis given by

$$\text{for } \mathfrak{g}_{247G} : \{2a, 2b, -b + c, 4d, -2d + 2e, 8f, -4f + 4g\}$$

yields the same structure table of

$$\mathfrak{g}_{247H_1} : ab = d, ac = e, ad = f, bd = f, be = g, cd = g, ce = -g.$$

On the other hand,

$$ab = d, ac = e, ad = f, bc = e, be = g, cd = g, ce = f.$$

is a real Lie algebra isomorphic to  $\mathfrak{g}_{247H}$  over  $\mathbb{C}$  and probably is the real Lie algebra that Gong intended to call  $\mathfrak{g}_{247H_1}$  (see item 5 below).

4. The structure table of  $\mathfrak{g}_{247G}$  is that of  $\mathfrak{g}(0)$  where

$$\begin{aligned} \mathfrak{g}(t) : ab = d, ac = e, ad = (1 + \frac{t^3}{2})f + \frac{t^3}{2}g, ae = (1 - \frac{t^3}{2})f - \frac{t^3}{2}g, \\ bd = f, be = g, cd = g, ce = f. \end{aligned}$$

If we rewrite the structure table of

$$\mathfrak{g}_{247H} : ab = d, ac = e, ad = f, bd = f, be = g, cd = g, ce = f$$

in the basis

$$\begin{aligned} \{2t^2a + (\frac{1}{2} - t^2)b + \frac{1}{2}c, \frac{1}{2}(1+t)b + \frac{1}{2}(1-t)c, \frac{1}{2}(1-t)b + \frac{1}{2}(1+t)c, \\ t^2(1+t)d + t^2(1-t)e, t^2(1-t)d + t^2(1+t)e, \\ t^2(1+t^2)f + t^2(1-t^2)g, t^2(1-t^2)f + t^2(1+t^2)g\}, \end{aligned}$$

then it coincides with the table of  $\mathfrak{g}(t)$  and this shows that  $\mathfrak{g}(t) \simeq \mathfrak{g}_{247H}$  for  $t \neq 0$ .

5. In [GR] it is claimed that  $\mathfrak{g}_{247K}$  is rigid in  $\mathcal{N}_{7,3}$ , which, according to us, is not true. The Lie algebra claimed to be rigid in [GR] is

$$ab = c, ac = d, ae = f, af = g, bc = d, be = f, ce = g, ef = d$$

and the change of basis given by  $\{b, -a+b, e, c, f, d, -g\}$  yields the structure table of

$$\mathfrak{g}_{247K} : ab = d, ac = e, ad = f, be = g, cd = g, ce = f.$$

This Lie algebra is  $\mathfrak{g}(0)$  where

$$\mathfrak{g}(t) : ab = d, ac = e, ad = f, bc = t^2e, be = g, cd = g, ce = f.$$

and the change of basis given by

$$\{-ia, -it^2(a-b), tc, t^2d, -ite, -it^2f, t^3g\}$$

yields the structure table of  $\mathfrak{g}_{247H}$  (which is rigid) showing that  $\mathfrak{g}(t) \simeq \mathfrak{g}_{247H}$  over  $\mathbb{C}$  for all  $t \neq 0$ . Since  $\mathfrak{g}_{247H} \not\simeq \mathfrak{g}_{247K}$  we obtain that the Lie algebra  $\mathfrak{g}_{247K}$  is not rigid.

6. As a byproduct of the previous analysis, we also observed  $\mathfrak{g}_{247P_1} \simeq \mathfrak{g}_{247F}$  over  $\mathbb{R}$  in contrast to what is claimed in [Go], pages 114–115. In this case

$$\mathfrak{g}_{247P_1} : ab = d, ac = e, bc = f, bd = g, ce = f$$

$$\mathfrak{g}_{247F} : ab = d, ac = e, bd = f, be = g, cd = g, ce = f,$$

and the change of basis given by

$$\mathfrak{g}_{247P_1} : a, b+e, c, d, e, f, g$$

$$\mathfrak{g}_{247F} : a, b-c, b+c, d-e, d+e, 2f+2g, 2f-2g$$

yields the same structure table.

## REFERENCES

- [Bu] D. Burde, *Degenerations of 7-dimensional nilpotent lie algebras*, Comm. in Algebra, **33**(2005), 1259–1277. DOI: 10.1081/AGB-200053956 4.3
- [Ca] R. Carles, *Weight systems for complex nilpotent Lie algebras and application to the varieties of Lie algebras*, Publ. Univ. Poitiers, 96 (1996). 4.3, 4.4
- [CSS] M. Crainic, F. Schätz and I. Struchiner, *A survey on stability and rigidity results for Lie algebras*, Indagationes Mathematicae, **25** (2014), 957–976. 1.1
- [CdGS] S. Cicalò, W. de Graaf, C. Schneider, *Six-dimensional nilpotent Lie algebras*, Linear Algebra and its Applications, **436** (2012), 163–189 3.3, 4.2
- [dG] W. de Graaf, *Classification of 6-dimensional nilpotent Lie algebras over fields of characteristic not 2*, J. of Algebra, Vol. **309**, 640–653 4.1
- [Go] M. P. Gong, *Classification of nilpotent Lie algebras of dimension 7 (over algebraically closed fields and  $\mathbb{R}$ )*, Ph.D. thesis, University of Waterloo, Waterloo, Canada, 1998. 4.4, 3, 6
- [GA] M. Goze and J. M. Ancochea Bermudez, *On the varieties of nilpotent Lie algebras of dimension 7 and 8*, J. of Pure and Applied Algebra **77** (1992) 131–140. 4.3
- [GR] M. Goze and E. Remm, *k-step nilpotent Lie algebras*, Georgian Math. J. Vol **22**, (2015) 219–234. 1.2, 5
- [GK] M. Goze and Y. Khakimdjani, *Nilpotent Lie algebras*, Mathematics and its Applications, **361**, Kluwer Academic Publishers Group, Dordrecht, (1996). Manuscripta Math. 84 (1994), 115–224. 1.1
- [GO] F. Grunewald, J. O’Halloran, *Varieties of nilpotent Lie algebras of dimension less than six*, J. Algebra **112** (1988), 31–325. 4.1
- [H] R. S. Hamilton, *The inverse function theorem of Nash and Moser*, Bull. Amer. Math. Soc., **7** (1982), 65–222. 1.1, 2.1, 2.1

- [Ma] L. Magnin, *Adjoint and Trivial Cohomology Tables for Indecomposable Nilpotent Lie Algebras of Dimension  $\leq 7$  over  $\mathbb{C}$* , eBook, 2nd Corrected Edition 2007, [http://math.u-bourgogne.fr/topology/magnin/public\\_html/Magnin2.ps](http://math.u-bourgogne.fr/topology/magnin/public_html/Magnin2.ps) 4.3, 4.4
- [Mo] J. Moser, *A rapidly convergent iteration method and non-linear differential equations*, Ann. Scuola Norm. Sup. Pisa (3) 20 (1966), 499-535. 1.1, 2.1
- [Na] J. Nash, *The embedding problem for Riemannian manifolds*, Ann. of Math. (2) 63 (1956), 20-63. 1.1, 2.1
- [NR] A. Nijenhuis and R. W. Richardson, Jr., *Cohomology and deformations in graded Lie algebras*, Bull. Amer. Math. Soc., **73** (1967), 175-179. 1.1
- [R] R. W. Richardson, Jr., *On the rigidity of semi-direct products of Lie algebras*, Pacific Journal of Mathematics, **22**, No. 2, (1967), 339-344. 1.2, 4.1
- [Se1] C. Seeley, *Degenerations of 6-dimensional nilpotent Lie algebras over  $\mathbb{C}$* , Comm. in Algebra 18 (1990), 3493-3505. 3.3, 4.2
- [Se2] C. Seeley, *7-dimensional nilpotent Lie algebras*, Transactions of the American Mathematical Society, vol. 335 (1993), 479-496. 4.3, 4.4
- [S] JP. Serre, *Lie Algebras and Lie Groups: 1964 Lectures given at Harvard University*, Lecture Notes in Mathematics, Springer, 2nd Edition. 1.1

CIEM-CONICET, FAMAF-UNIVERSIDAD NACIONAL DE C RDOBA  
 E-mail address: [brega@famaf.unc.edu.ar](mailto:brega@famaf.unc.edu.ar)

CIEM-CONICET, FAMAF-UNIVERSIDAD NACIONAL DE C RDOBA  
 E-mail address: [cagliero@famaf.unc.edu.ar](mailto:cagliero@famaf.unc.edu.ar)

CIEM-CONICET, FAMAF-UNIVERSIDAD NACIONAL DE C RDOBA  
 E-mail address: [aeco03@yahoo.com](mailto:aeco03@yahoo.com)